

# Blow-up Rate Estimates for Parabolic Equations

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## Abstract

We consider the blow-up sets and the upper blow-up rate estimates for two parabolic problems defined in a ball  $B_R$  in  $R^n$ ; firstly, the semilinear heat equation  $u_t = \Delta u + e^{u^p}$  subject to the zero Dirichlet boundary conditions, secondly, the problem of the heat equation  $u_t = \Delta u$  with the Neumann boundary condition  $\frac{\partial u}{\partial \eta} = e^{u^p}$  on  $\partial B_R \times (0, T)$ , where  $p > 1$ ,  $\eta$  is the outward normal.

## 1 Introduction

In this paper, we study two problems of parabolic equations:

$$\left. \begin{aligned} u_t &= \Delta u + e^{u^p}, & (x, t) &\in B_R \times (0, T), \\ u(x, t) &= 0, & (x, t) &\in \partial B_R \times (0, T), \\ u(x, 0) &= u_0(x), & x &\in B_R, \end{aligned} \right\} \quad (1.1)$$

and

$$\left. \begin{aligned} u_t &= \Delta u, & (x, t) &\in B_R \times (0, T), \\ \frac{\partial u}{\partial \eta} &= e^{u^p}, & (x, t) &\in \partial B_R \times (0, T), \\ u(x, 0) &= u_0(x), & x &\in B_R, \end{aligned} \right\} \quad (1.2)$$

where  $p > 1$ ,  $B_R$  is a ball in  $R^n$ ,  $\eta$  is the outward normal,  $u_0$  is smooth, nonzero, nonnegative, radially symmetric, moreover, for problem (1.1) it is further required to be nonincreasing radial function, vanishing on  $\partial B_R$  and satisfies the following condition

$$\Delta u_0(x) + e^{u_0^p(x)} \geq 0, \quad x \in B_R, \quad (1.3)$$

while for problem (1.2), it is required to satisfy the following conditions

$$\frac{\partial u_0}{\partial \eta} = e^{u_0^p}, \quad x \in \partial B_R, \quad (1.4)$$

$$\Delta u_0 \geq 0, \quad x \in \overline{B_R}. \quad (1.5)$$

Blow-up phenomena for reaction-diffusion problems in bounded domain have been studied for the first time in [10] by Kaplan, he showed that, if the convex source terms  $f = f(u)$  satisfying the condition

$$\int_U^\infty \frac{du}{f(u)} < \infty, \quad U \geq 1, \quad (1.6)$$

then diffusion cannot prevent blow-up when the initial state is large enough.

The problem of semilinear parabolic equation defined in a ball, has been introduced in [7, 12, 15, 16], for instance, in [7] Friedman and McLeod have studied the zero Dirichlet problem of the semilinear heat equation:

$$u_t = \Delta u + f(u), \quad \text{in } B_R \times (0, T), \quad (1.7)$$

under fairly general assumptions on  $u_0$  (nonincreasing radial function, vanishing on  $\partial B_R$ ). They have considered the two special cases:

$$u_t = \Delta u + u^p, \quad p > 1, \quad (1.8)$$

$$u_t = \Delta u + e^u. \quad (1.9)$$

For equation (1.8), they showed that for any  $\alpha > 2/(p-1)$ , the upper pointwise estimate takes the following form

$$u(x, t) \leq C|x|^{-\alpha}, \quad x \in B_R \setminus \{0\} \times (0, T),$$

which shows that the only possible blow-up point is  $x = 0$ . Moreover, under an additional assumption of monotonicity in time (1.3), the corresponding lower estimate on the blow-up can be established (see[15]) as follows

$$u(x, T) \geq C|x|^{-2/(p-1)}, \quad x \in B_{R^*} \setminus \{0\},$$

for some  $R^* \leq R$ ,  $C > 0$ . On the other hand, it has been shown in [7] that the upper (lower) blow-up rate estimates take the following form

$$c(T-t)^{-1/(p-1)} \leq u(0, t) \leq C(T-t)^{-1/(p-1)}, \quad t \in (0, T).$$

For the second case, (1.9), Friedman and McLeod showed similar results, they proved that the point  $x = 0$  is the only blow-up point, and that due to the upper pointwise estimate, which takes the following form

$$u(x, t) \leq \log C + \frac{2}{\alpha} \log\left(\frac{1}{x}\right), \quad (x, t) \in B_R \setminus \{0\} \times (0, T),$$

where  $0 < \alpha < 1$ ,  $C > 0$ . Moreover, the upper (lower) blow-up rate estimate takes the following form

$$\log c - \log(T-t) \leq u(0, t) \leq \log C - \log(T-t), \quad t \in (0, T). \quad (1.10)$$

The aim of section two, is to show that the results of Friedman and McLeod hold true for problem (1.1). On other words, we prove that  $x = 0$  the only possible blow-up point for this problem. Furthermore, we show that the upper blow-up rate estimate takes the following form

$$u(0, t) \leq \log C - \frac{1}{p} \log(T - t), \quad t \in (0, T).$$

The problem of the heat equation defined in a ball  $B_R$  with a nonlinear Neumann boundary condition,  $\frac{\partial u}{\partial \eta} = f(u)$  on  $\partial B_R \times (0, T)$ , has been introduced in [1, 3, 4, 8], for instance, in [8] it has been shown that if  $f$  is nondecreasing and  $1/f$  is integrable at infinity for  $u > 0$ , then the blow-up occurs in finite time for any positive initial data  $u_0$  (not necessarily radial), moreover, if  $f$  is  $C^2(0, \infty)$ , increasing and convex in  $(0, \infty)$ , then blow-up occurs only on the boundary.

For the special case, where  $f(u) = u^p$ , it has been proved in [3] that for any  $u_0$ , the finite time blow-up occurs where  $p > 1$ , and it occurs only on the boundary. Moreover, it has been shown in [4, 9] that the upper (lower) blow-up rate estimate take the following form

$$C_1(T - t)^{\frac{-1}{2(p-1)}} \leq \max_{x \in \overline{B}_R} u(x, t) \leq C_2(T - t)^{\frac{-1}{2(p-1)}}, \quad t \in (0, T).$$

In [1], it has been considered the second special case, where,  $f(u) = e^u$ , in one dimensional space defined in the domain  $(0, 1) \times (0, T)$ , it has been proved that every positive solution blows up in finite time and the blow-up occurs only on the boundary ( $x = 1$ ) and the upper (lower) blow-up rate estimates take the following forms

$$C_1(T - t)^{-1/2} \leq e^{u(1, t)} \leq C_2(T - t)^{-1/2}, \quad 0 < t < T.$$

Section three concerned with the blow-up solutions of problem (1.2), we prove that the upper blow-up rate estimate takes the following form

$$\max_{\overline{B}_R} u(x, t) \leq \log C - \frac{1}{2p} \log(T - t), \quad 0 < t < T.$$

## 2 Problem (1.1)

### 2.1 Preliminaries

Since  $f(u) = e^{u^p}$  is  $C^1([0, \infty))$  function, the existence and uniqueness of local classical solutions to problem (1.1) are well known, see [5, 11]. On the other hand, since the function  $f$  is convex on  $(0, \infty)$  and satisfies the condition (1.6),

therefore, the solutions of problem (1.1) blow up in finite time for large initial function.

The next lemma shows some properties of the solutions of problem (1.1). We denote for simplicity  $u(r, t) = u(x, t)$ .

**Lemma 2.1.** *Let  $u$  be a classical solution of (1.1). Then*

- (i)  $u(x, t)$  is positive and radial,  $u_r \leq 0$  in  $[0, R) \times (0, T)$ . Moreover,  $u_r < 0$  in  $(0, R] \times (0, T)$ .
- (ii)  $u_t > 0$ ,  $(x, t) \in B_R \times (0, T)$ .

## 2.2 Pointwise Estimates

This subsection considers the pointwise estimate to the solutions of problem (1.1), which shows that the blow-up cannot occur if  $x$  is not equal zero. In order to prove that, we need first to recall the following lemma, which has been proved by Friedman and McLeod in [7].

**Lemma 2.2.** *Let  $u$  be a blow-up solution of the zero Dirichlet problem of (1.7) with  $u_0$  is nonzero, nonincreasing radial function vanishing on  $\partial B_R$ . Also suppose that*

$$u_{0r}(r) \leq -\delta r, \quad \text{for } 0 < r \leq R, \quad \text{where } \delta > 0. \quad (2.1)$$

*If there exist  $F \in C^2(0, \infty) \cap C^1([0, \infty))$ , such that  $F$  is positive in  $(0, \infty)$  and satisfies*

$$\int_s^\infty \frac{du}{F(u)} < \infty, \quad F', F'' \geq 0 \quad \text{in } (0, \infty). \quad (2.2)$$

*Also if it satisfies with  $f$  the following condition,*

$$f'F - fF' \geq 2\varepsilon FF' \quad \text{in } (0, \infty), \quad (2.3)$$

*then the function  $J = r^{n-1}u_r + \varepsilon r^n F(u)$  is nonpositive in  $B_R \times (0, T)$  for some  $\varepsilon > 0$ .*

**Theorem 2.3.** *Let  $u$  be a blow-up solution of problem (1.1). Also suppose that  $u_0$  satisfies (2.1). Then  $x = 0$  is the only blow-up point.*

*Proof.* Let

$$F(u) = e^{\delta u^p}, \quad 0 < \delta < 1.$$

It is clear that  $F$  satisfies (2.2). The next aim is to show that the inequality (2.3) holds.

A direct calculation shows

$$\begin{aligned} f'(u)F(u) - f(u)F'(u) &= pu^{p-1}e^{(1+\delta)u^p} - \delta pu^{p-1}e^{(1+\delta)u^p} \\ &= pu^{p-1}e^{(1+\delta)u^p}[1 - \delta]. \end{aligned} \quad (2.4)$$

On the other hand,

$$2\varepsilon F(u)F'(u) = 2\varepsilon \delta p u^{p-1} e^{2\delta u^p}. \quad (2.5)$$

From (2.4), (2.5) it is clear that (2.3) holds true provided  $\varepsilon, \delta$  are small enough.

Thus

$$J = r^{n-1} u_r + \varepsilon r^n e^{\delta u^p} \leq 0, \quad (r, t) \in (0, R) \times (0, T),$$

or

$$-\frac{u_r}{e^{\delta u^p}} \geq \varepsilon r. \quad (2.6)$$

Let  $G(s) = \int_s^\infty \frac{du}{e^{\delta u^p}}$ .

It is clear that

$$\frac{d}{dr} G(u(r, t)) = \frac{d}{dr} \int_u^\infty \frac{du}{e^{\delta u^p}} = -\frac{d}{dr} \int_\infty^u \frac{du}{e^{\delta u^p}} = -\frac{d}{du} \int_\infty^u \frac{u_r}{e^{\delta u^p}} du = -\frac{u_r}{e^{\delta u^p}}.$$

Thus, by (2.6), we obtain

$$G(u(r, t))_r \geq \varepsilon r.$$

Now, integrate the last equation from 0 to  $r$

$$G(u(r, t)) - G(u(0, t)) \geq \frac{1}{2} \varepsilon r^2.$$

It follows

$$G(u(r, t)) \geq \frac{1}{2} \varepsilon r^2. \quad (2.7)$$

If for some  $r > 0$ ,  $u(r, t) \rightarrow \infty$ , as  $t \rightarrow T$ , then  $G(u(r, t)) \rightarrow 0$ , as  $t \rightarrow T$ , a contradiction to (2.7).  $\square$

## 2.3 Blow-up Rate Estimate

The following theorem considers the upper bounds of the blow-up rate for problem (1.1).

**Theorem 2.4.** *Let  $u$  be a solution of (1.1), which blows up at only  $x = 0$ , in finite time  $T$ . Then there exists a positive constant  $C$  such that*

$$u(0, t) \leq \log C - \frac{1}{p} \log(T - t), \quad t \in (0, T). \quad (2.8)$$

*Proof.* Define the function  $F$  as follows,

$$F(x, t) = u_t - \alpha f(u), \quad (x, t) \in B_R \times (0, T),$$

where  $f(u) = e^{u^p}$ ,  $\alpha > 0$ .

A direct calculation shows

$$\begin{aligned}
F_t - \Delta F &= u_{tt} - \alpha f' u_t - \Delta u_t + \alpha \Delta f(u), \\
&= u_{tt} - \Delta u_t - \alpha f' [u_t - \Delta u] + \alpha |\nabla u|^2 f'', \\
&= f' u_t - \alpha f' f(u) + \alpha |\nabla u|^2 f''.
\end{aligned}$$

Thus

$$F_t - \Delta F - f'(u)F = \alpha |\nabla u|^2 f'' \geq 0, \quad (x, t) \in B_R \times (0, T), \quad (2.9)$$

due to  $f''(u) > 0$ , for  $u$  in  $(0, \infty)$ .

Since,  $f'$  is continuous, therefore,  $f'(u)$  is bounded in  $\overline{B_R} \times [0, t]$ , for  $t < T$ .

By Lemma 2.1,  $u_t(x, t) > 0$ , in  $B_R \times (0, T)$ , and since  $u$  blows up at  $x = 0$ , therefore, there exist  $k > 0$ ,  $\varepsilon \in (0, R)$ ,  $\tau \in (0, T)$  such that

$$u_t(x, t) \geq k, \quad (x, t) \in \overline{B_\varepsilon} \times [\tau, T).$$

Also, we can find  $\alpha > 0$  such that  $u_t(x, \tau) \geq \alpha f(u(x, \tau))$ , for  $x \in B_\varepsilon$ . Thus

$$F(x, \tau) \geq 0 \quad \text{for } x \in B_\varepsilon. \quad (2.10)$$

On the other hand, because of  $u$  blows up at only  $x = 0$ , there exists  $C_0 > 0$  such that

$$f(u(x, t)) \leq C_0 < \infty, \quad \text{in } \partial B_\varepsilon \times (0, T),$$

If we choose  $\alpha$  is small enough such that  $k \geq \alpha C_0$ , then we get

$$F(x, t) \geq 0, \quad (x, t) \in \partial B_\varepsilon \times [\tau, T), \quad (2.11)$$

By (2.9), (2.10), (2.11) and maximum principle [13], it follows that

$$F(x, t) \geq 0, \quad (x, t) \in \overline{B_\varepsilon} \times (\tau, T).$$

Thus

$$u_t(0, t) \geq \alpha e^{u^p(0, t)}, \quad \text{for } \tau \leq t < T. \quad (2.12)$$

Since  $u$  is increasing in time and blows at  $T$ , there exist  $\tau^* \leq \tau$  such that

$$u(0, t) \geq p^{\frac{1}{(p-1)}} \quad \text{for } \tau^* \leq t < T,$$

provided  $\tau$  is close enough to  $T$ , which leads to

$$e^{u^p(0, t)} \geq e^{pu(0, t)}, \quad \tau^* \leq t < T. \quad (2.13)$$

From (2.12), (2.13), it follows that

$$u_t(0, t) \geq \alpha e^{pu(0, t)}, \quad \text{for } \tau \leq t < T. \quad (2.14)$$

Integrate (2.14) from  $t$  to  $T$

$$\int_t^T u_t(0, t) e^{-pu(0, t)} \geq \alpha(T - t).$$

Thus

$$-\frac{1}{p} e^{-pu(0, t)} \Big|_t^T \geq \alpha(T - t). \quad (2.15)$$

Since

$$u(0, t) \rightarrow \infty, \quad e^{-pu(0, t)} \rightarrow 0, \quad \text{as } t \rightarrow T,$$

therefore, (2.15) becomes

$$\frac{1}{e^{pu(0, t)}} \geq p\alpha(T - t).$$

Thus

$$e^{pu(0, t)}(T - t) \leq C^*, \quad C^* = 1/(p\alpha), \quad t \in [\tau, T)$$

Therefore, there exist a positive constant  $C$  such that

$$u(0, t) \leq \log C - \frac{1}{p} \log(T - t), \quad t \in (0, T).$$

□

## 3 Problem (1.2)

### 3.1 Preliminaries

The local existence of the unique classical solutions to problem (1.2) is well known (see [8]). On the other hand, since  $f(u) = e^{u^p}$  is  $C^2(0, \infty)$ , increasing, positive function in  $(0, \infty)$  and  $1/f$  is integrable at infinity for  $u > 0$ , moreover,  $f$  is convex ( $f''(u) > 0, \forall u > 0$ ). Therefore, according to the result of [8], the solutions of problem (1.2) blow up infinite time and the blow-up occurs only on the boundary.

The following lemma shows some properties of the solutions of problem (1.2). We denote for simplicity  $u(r, t) = u(x, t)$ .

**Lemma 3.1.** *Let  $u$  be a classical unique solution to problem (1.2). Then*

- (i)  $u > 0$ , radial on  $\overline{B}_R \times (0, T)$ . Moreover,  $u_r \geq 0$ , in  $[0, R] \times [0, T)$ .
- (ii)  $u_t > 0$  in  $\overline{B}_R \times (0, T)$ . Moreover, if  $\Delta u_0 \geq a > 0$ , in  $\overline{B}_R$ , then  $u_t \geq a$ , in  $\overline{B}_R \times [0, T)$ .

### 3.2 Blow-up Rate Estimate

The following theorem considers the upper blow-up rate estimate of problem (1.2).

**Theorem 3.2.** *Let  $u$  be a blow-up solution to (1.2), where  $\Delta u_0 \geq a > 0$  in  $\overline{B}_R$ ,  $T$  is the blow-up time. Then there exists a positive constant  $C$  such that*

$$\max_{\overline{B}_R} u(x, t) \leq \log C - \frac{1}{2p} \log(T - t), \quad 0 < t < T. \quad (3.1)$$

*Proof.* We follow the idea of [1], consider the function

$$F(x, t) = u_t(r, t) - \varepsilon u_r^2(r, t), \quad (x, t) \in B_R \times (0, T).$$

By a straightforward calculation

$$F_t - \Delta F = 2\varepsilon \left( \frac{n-1}{r^2} u_r^2 + u_{rr}^2 \right) \geq 0.$$

Since  $\Delta u_0 \geq a > 0$ , and  $u_{0r} \in C(\overline{B}_R)$ ,

$$F(x, 0) = \Delta u_0(r) - \varepsilon u_{0r}^2(r) \geq 0, \quad x \in B_R.$$

provided  $\varepsilon$  is small enough.

Moreover,

$$\begin{aligned} \frac{\partial F}{\partial \eta}|_{x \in S_R} &= u_{rt}(R, t) - 2\varepsilon u_r(R, t) u_{rr}(R, t) \\ &= (e^{u^p(R, t)})_t - 2\varepsilon e^{u^p(R, t)} (u_t(R, t) - \frac{n-1}{r} u_r(R, t)) \\ &\geq (p[u(R, t)]^{p-1} - 2\varepsilon) e^{u^p(R, t)} u_t(R, t). \end{aligned}$$

Since

$$u_t > 0, \quad \text{on } \overline{B}_R \times (0, T).$$

Thus

$$\frac{\partial F}{\partial \eta}|_{x \in S_R} \geq 0, \quad t \in (0, T)$$

provided

$$\varepsilon \leq \frac{p[u_0(R)]^{p-1}}{2}.$$

From the comparison principle [13], it follows that

$$F(x, t) \geq 0, \quad \text{in } \overline{B}_R \times (0, T),$$



in particular  $F(x, t) \geq 0$ , for  $|x| = R$ , that is

$$u_t(R, t) \geq \varepsilon u_r^2(R, t) = \varepsilon e^{2u^p(R, t)}, \quad t \in (0, T).$$

Since  $u$  is increasing in time and blows at  $T$ , there exist  $\tau \leq T$  such that

$$u(R, t) \geq p^{\frac{1}{(p-1)}} \quad \text{for } \tau \leq t < T,$$

which leads to

$$u_t(R, t) \geq \varepsilon e^{2pu(R, t)}, \quad t \in [\tau, T).$$

By integration the above inequality from  $t$  to  $T$ , it follows that

$$\int_t^T u_t e^{-2pu(R, t)} \geq \varepsilon(T - t).$$

So

$$-\frac{1}{2p} e^{-2pu(R, t)} \Big|_t^T \geq \varepsilon(T - t). \quad (3.2)$$

Since

$$u(R, t) \rightarrow \infty, \quad e^{-pu(R, t)} \rightarrow 0 \quad \text{as } t \rightarrow T,$$

the inequality (3.2) becomes

$$\frac{1}{e^{pu(R, t)}} \geq (2p\varepsilon(T - t))^{1/2},$$

which means

$$(T - t)^{1/2} e^{pu(R, t)} \leq \frac{1}{\sqrt{2p\varepsilon}},$$

Therefore, there exist a positive constant  $C$  such that

$$\max_{\overline{B}_R} u(x, t) \leq \log C - \frac{1}{2p} \log(T - t), \quad 0 < t < T.$$

□

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